

A short note on Merlin-Arthur protocols for subset sum

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In the SUBSET SUM problem we are given positive integers w_1, \dots, w_n along with a target integer t . The task is to determine whether there exists a subset $X \subseteq \{1, \dots, n\}$ such that $\sum_{i \in X} w_i = t$. We refer to such an X as a *solution*. The goal of this short note is to prove the following:

Theorem 1. *For a given SUBSET SUM instance w_1, \dots, w_n, t there is a proof of size¹ $O^*(\sqrt{t})$ of what the number of solutions is that can be constructed in $O^*(t)$ time and can be probabilistically verified in time $O^*(\sqrt{t})$ with at most constant error probability.*

In complexity theory, a proof system as above is commonly referred to as a *Merlin-Arthur protocol*. These protocols very recently received attention [6, 3] in the exponential time setting: Williams [6] gives very generic Merlin-Arthur protocols with verifiers more efficient than best known exponential time algorithms. By no means we claim this note is innovative in any way: both our work and [6] have a similar flavor and it is likely that Theorem 1 is also obtainable via a clever application of [6, Theorem 1.1], or at the very least our observation will be less surprising given [6, 3].

Proof of Theorem 1. The proof system is outlined in Algorithm 1. Note that both the prover and the verifier algorithms are simple modifications of the well known pseudo-polynomial time dynamic programming algorithm for SUBSET SUM usually attributed to Bellman.

Algorithm P(w_1, \dots, w_n, t)

Prove that the number of solutions is c_t .

Output: Prime $p = \Theta(\sqrt{nt})$, integer c_i for $i \leq nt, i \equiv_p t$ with $c_i = |\{X \subseteq [n] : w(X) = i\}|$.

1: Initiate $T[0, 0] = 1$ and $T[0, i] = 0$ for $0 < i \leq nt$.

2: **for** $j = 1 \rightarrow n$ **do**

3: **for** $i = 1 \rightarrow n \cdot t$ **do**

4: **if** $i < w_j$ **then** $T[j, i] \leftarrow T[j - 1, i]$ **else** $T[j, i] \leftarrow T[j - 1, i] + T[j - 1, i - w_j]$.

5: Pick smallest prime p satisfying $2\sqrt{nt} < p < 4\sqrt{nt}$.

Brute-force and [1] is within time budget.

6: **for** $i \leq n \cdot t$ such that $i \equiv_p t$ **do**

7: $c_i \leftarrow T[n, i]$.

Algorithm V($w_1, \dots, w_n, t, p, \{c_i\}$)

Verify the proof for the number of solutions.

Output: Yes if the proof is as output by P, no with constant probability if the proof is ‘incorrect’.

1: Pick a prime q satisfying $2^{nt} < q < 2^{nt+1}$.

Sample and check primality until success.

2: Pick a random element $r \in \mathbb{Z}_q$.

3: Initiate $T'[0, 0] = 1$ and $T'[0, i] = 0$ for $i \in \mathbb{Z}_p$.

4: **for** $j = 1 \rightarrow n$ **do**

5: **for** $i = 1 \rightarrow n \cdot t$ **do**

6: $T'[j, i] \leftarrow (T'[j - 1, i] + r^{w_j} \cdot T'[j - 1, (i - w_j) \% p]) \% q$.

7: Compute $\sum_i c_i r^i \% q$.

8: **if** $\sum_i c_i r^i \equiv_q T'[n, t \% p]$ **then return yes else return no**.

Algorithm 1: The prover-verifier protocol.

Recall the Prime Number Theorem, that is used in Lines 5 and 1:

Claim 2. *For large enough integers b there exist at least $2^b/b$ prime numbers p in the interval $2^b < p < 2^{b+1}$.*

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¹ $O^*(\cdot)$ omits factors polynomial in the input size.

By this result, we may sample $2^{nt} < q < 2^{n+1}t$ and check primality until success and declare YES after $n + \lg(t)$ unsuccessful tries (which happens with probability at most $1/4$).

For the verification probability, it is straightforward to see that $T[n, i]$ equals $|\{X \subseteq [n] : w(X) = i\}|$. Similarly, when considering r as indeterminate, we see that $T'[n, t]$ is a polynomial in $\mathbb{Z}_q[r]$ satisfying

$$T'[n, t] \equiv_q \sum_{\substack{i \leq nt \\ i \equiv_p t}} |\{X \subseteq [n] : w(X) = i\}| r^i.$$

Also note that $|\{X \subseteq [n] : w(X) = j\}| < q$. Thus we see that if the proof is correctly constructed as in P, $\sum_i c_i r^i \bmod q$ and $T'[n, t]$ are equivalent polynomials and V accepts. On the other hand, if the proof is incorrect, e.g. $c_i \neq T[n, i]$ for some i , we see that $(\sum_i c_i r^i \bmod q) - T'[n, t]$ is not the zero polynomial and has degree at most $n \cdot t$. Thus, it has at most $n \cdot t$ roots in \mathbb{Z}_q , and the probability that r is one of these roots is $nt/q < 1/4$. If r is not a root of the difference polynomial, the two evaluations necessarily differ and the verifier rejects. Thus in total the verifier does not reject an incorrect proof with probability at most $1/2$.

For the running time of P, note that all involved integers are at most 2^n and the running time bound follows directly. The running time of V follows similarly by using taking modulus at each powering step at Line 6. \square

Let us note that the above theorem also gives rise to similar results for parity versions of Set Partition, Set Cover, Hitting Set and CNF-Sat via the reductions of [5, 4] (since the reduction in [4, Theorem 4.9] from Subset Sum/ m to Set Partition is parsimonious). But all of this was also shown in a stronger form in [6, 3].

Also [2, Theorem 1.6], in combination with the above theorem gives an $O^*(2^{.499n})$ -sized proof that can be probabilistically verified in time $O^*(2^{.499n})$ for SUBSET SUM. However, combining [6, Theorem A.1] with the methods from [2] gives a more efficient proof system leading to a proof of size $O^*(2^{.3113n})$ that can be verified in time $O^*(2^{.3113n})$.

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